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INVESTIGATION OF THE STRESS STATE OF A COMPOSITE IN THE FORM OF A LAYER AND A HALF-SPACE WITH A CYLINDRICAL TUBE AT GIVEN STRESSES ON THE BOUNDARY SURFACES

Abstract. A solution to the spatial problem of the theory of elasticity is proposed for a composite in the form of a half-space with a longitudinal thick-walled circular cylindrical tube and a layer rigidly attached to the surface of the half-space. Layer, half-space and pipe - elastic homogeneous isotropic materials different from each other.

The stresses are set on the free surface of the layer and the inner surface of the pipe. At the boundary of the layer and half-space, as well as at the boundary of half-space and the outer surface of the pipe, the matching conditions are coupling. It is necessary to evaluate the stress state of a given composite.

The solution of the spatial problem of the theory of elasticity is obtained on the basis of the generalized Fourier method in cylindrical coordinates associated with the pipe and Cartesian coordinates associated with the layer and half-space. Satisfying the boundary and coupling conditions, we obtain infinite systems of linear algebraic equations that are solved by the reduction method. As a result, displacements and stresses were obtained at various points of the layer, half-space, and pipe.

Keywords: thick-walled pipe in half-space, composite, coupling conditions, generalized Fourier method

Introduction

When designing building structures, underground structures and communications, as well as in mechanical engineering, one has to deal with design schemes in which a composite medium is present. However, effective methods for calculating structures with several boundary surfaces (more than three) are practically absent.

For such problems, the generalized Fourier method is used, which was supplemented by the theorems of addition of basic solutions [1].

Based on this method, problems are solved for a space with cylindrical cavities and various boundary conditions [2], half-spaces with a cylindrical cavity or inclusion [3–8], for a cylinder with cylindrical cavities or inclusions [9], for a layer with a cylindrical cavity, inclusion or tube [10–13].

Formulation of the Problem

An elastic homogeneous layer of height h_1 is rigidly connected with an elastic homogeneous half-space. In a half-space, parallel to its surface, there is a circular cylindrical thick-walled pipe with an outer radius R_1, and an inner one - R_2.

We will consider the pipe in a cylindrical coordinate system (ρ, φ, z), the half-space in the Cartesian coordinate system (x_2, y_2, z_2), which is identically oriented and combined with the coordinate system of the pipe. The half-space boundary is located at $y_2=h_2$. The layer will be considered in the Cartesian coordinate system (x_1, y_1, z_1) located on the lower surface of the layer (the interface between the half-space and, accordingly, shifted relative to the half-space coordinate system by $y_2=h_2$).

It is necessary to find a solution to the Lame equation $\Delta\vec{U}_j + \left(1-2\sigma_j\right)^{-1}\nabla div\vec{U}_j = 0$, where σ_j – Poisson's ratio of the layer (j=1), half spaces (j=2) or pipes (j=3).

Stresses are set on the upper boundary of the layer $F_1\vec{U}_1(x,z)_{|y_1=h_1} = \vec{F}_h^0(x,z)$, the stresses on the inner surface of the pipe $F_3\vec{U}_3(\phi,z)_{|\rho=R} = \vec{F}_R^0(\phi,z)$, where $\vec{U}_1$ — displacement in the layer; $\vec{U}_3$— displacement in the pipe;

$F_j\vec{U}_j\big| = 2G_j[\frac{\sigma_j}{1-2\sigma_j}\vec{n}\,div\,U_j + \frac{\partial}{\partial n}\vec{U}_j + \frac{1}{2}(\vec{n}\times rot\vec{U}_j)]$;

$G_j = \frac{E_j}{2(1+\sigma_j)}$; σ_j, E_j – Poisson's ratio and modulus of elasticity of the layer (j = 1), half-space (j = 2) or pipe (j = 3);

$$\vec{F}_h^0(x_1,z_1) = \tau_{yx}^{(h)}\vec{e}_1^{(1)} + \sigma_y^{(h)}\vec{e}_2^{(1)} + \tau_{yz}^{(h)}\vec{e}_3^{(1)},$$
$$\vec{F}_R^0(\phi,z) = \sigma_\rho^{(R)}\vec{e}_1^{(2)} + \tau_{\rho\phi}^{(R)}\vec{e}_2^{(2)} + \tau_{\rho z}^{(R)}\vec{e}_3^{(2)} \quad (1)$$

are known functions; $\vec{e}_j^{(k)}$, j = 1, 2, 3 – are the unit vectors of the Cartesian (k = 1) and cylindrical (k = 2) coordinate systems.

On the boundary of the layer and half-space, coupling conditions are given

$$\vec{U}_1\big|_{y_1=0} = \vec{U}_2\big|_{y_2=h_2}, \quad (2)$$
$$F_1\vec{U}_1\big|_{y_1=0} = F_2\vec{U}_2\big|_{y_2=h_2}, \quad (3)$$

at the boundary of the half-space and the pipe, the coupling conditions are given

$$\vec{U}_2(\phi,z)_{|\rho=R_1} = \vec{U}_3(\phi,z)_{|\rho=R_1}, \quad (4)$$
$$F\vec{U}_2(\phi,z)_{|\rho=R_1} = F\vec{U}_3(\phi,z)_{|\rho=R_1}, \quad (5)$$

where $\vec{U}_2$— - displacement in half space.

All known vectors and functions will be considered as fast falling to zero at great distances from the origin of the coordinate z for the tube and the coordinates x and z for the boundaries of the layer.

Solving the Problem

We take the basic solutions of the Lame equation in the form [1]

$$\vec{u}_k^{\pm}(x,y,z;\lambda,\mu) = N_k^{(d)} e^{i(\lambda z+\mu x)\pm\gamma y};$$
$$\vec{R}_{k,m}(\rho,\phi,z;\lambda) = N_k^{(p)} I_m(\lambda\rho) e^{i(\lambda z+m\phi)};$$
$$\vec{S}_{k,m}(\rho,\phi,z;\lambda) = N_k^{(p)}\left[(sign\,\lambda)^m K_m(|\lambda|\rho)\cdot e^{i(\lambda z+m\phi)}\right]; k=1,2,3; \tag{6}$$

$$N_1^{(d)} = \frac{1}{\lambda}\nabla;\ N_2^{(d)} = \frac{4}{\lambda}(\sigma-1)\vec{e}_2^{(1)} + \frac{1}{\lambda}\nabla(y\,\cdot);\ N_3^{(d)} = \frac{i}{\lambda}rot(\vec{e}_3^{(1)}\,\cdot);\ N_1^{(p)} = \frac{1}{\lambda}\nabla;$$
$$N_2^{(p)} = \frac{1}{\lambda}\left[\nabla\left(\rho\frac{\partial}{\partial\rho}\right) + 4(\sigma-1)\left(\nabla - \vec{e}_3^{(2)}\frac{\partial}{\partial z}\right)\right];\ N_3^{(p)} = \frac{i}{\lambda}rot(\vec{e}_3^{(2)}\,\cdot);$$
$$\gamma = \sqrt{\lambda^2+\mu^2},\ -\infty<\lambda,\mu<\infty,$$

where $I_m(x)$, $K_m(x)$ – are the modified Bessel functions; $\vec{R}_{k,m}$, $\vec{S}_{k,m}$, k=1, 2, 3 – are, respectively, the internal and external solutions to the Lamé equation for the cylinder; $\vec{u}_k^{(-)}$, $\vec{u}_k^{(+)}$ – are the solutions to the Lamé equation for the layer and half-space

The solution to the problem will be presented in the form

$$\vec{U}_1 = \sum_{k=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(H_k^{(1)}(\lambda,\mu)\cdot\vec{u}_k^{(+)}(x_1,y_1,z_1;\lambda,\mu;\sigma_1) +$$
$$+\tilde{H}_k^{(1)}(\lambda,\mu)\cdot\vec{u}_k^{(-)}(x_1,y_1,z\ \ ;\lambda,\mu;\sigma_1)\Big)\,d\mu d\lambda, \tag{7}$$

$$\vec{U}_2 = \sum_{k=1}^{3}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}B_{k,m}(\lambda)\cdot\vec{S}_{k,m}(\rho,\phi,z;\lambda;\sigma_1)d\lambda +$$
$$+\sum_{k=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(H_k^{(2)}(\lambda,\mu)\cdot\vec{u}_k^{(+)}(x_2,y_2,z_2;\lambda,\mu;\sigma_2)\Big)\,d\mu d\lambda, \tag{8}$$

$$\vec{U}_3 = \sum_{k=1}^{3}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}A_{k,m}(\lambda)\cdot\vec{R}_{k,m}(\rho,\phi,z;\lambda) + \tilde{A}_{k,m}(\lambda)\cdot\vec{S}_{k,m}(\rho,\phi,z;\lambda)d\lambda, \tag{9}$$

where $\vec{S}_{k,m}(\rho,\phi,z;\lambda)$, $\vec{R}_{k,m}(\rho,\phi,z;\lambda)$ $\vec{u}_k^{(+)}(x,y,z;\lambda,\mu)$ and $\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)$ – are the basic solutions given by formulas (6), and the unknown functions $H_k^{(1)}(\lambda,\mu)$, $\tilde{H}_k^{(1)}(\lambda,\mu)$, $B_{k,m}(\lambda)$, $H_k^{(2)}(\lambda,\mu)$, $A_{k,m}(\lambda)$ and $\tilde{A}_{k,m}(\lambda)$ must be found from boundary conditions (1) and coupling conditions (2 – 5).

To transfer the main solutions between coordinate systems, we use the formulas [11].

To fulfill the boundary conditions at the upper boundary of the layer, we find the stresses for (7) and, for y_1=h_1, we equate the given $\vec{F}_h^0(x_1,z_1)$ one represented by the double Fourier integral. So we get three equations (one for each projection) with six unknowns $H_k^{(1)}(\lambda,\mu)$, $\tilde{H}_k^{(1)}(\lambda,\mu)$.

To satisfy the conjugation conditions at the boundary of the layer and half-space in displacements, we substitute the right-hand sides (7) and (8) in (2). In this case, writing down expression $\vec{U}_2(x_2,z_2)_{|y_2=0}$, it is necessary to use the formulas for the transition from solutions $\vec{S}_{k,m}$ of the cylinder to solutions $\vec{u}_k^{(-)}$ [12, formula (7)]. In a similar way, we can write three additional equations for stresses (3).

So we get nine infinite systems of equations with unknown functions $H_k^{(1)}(\lambda,\mu)$, $\tilde{H}_k^{(1)}(\lambda,\mu)$, $H_k^{(2)}(\lambda,\mu)$ and $B_{k,m}(\lambda)$.

The determinant Δ of this system has the form $\Delta = -64\cdot\gamma^9\cdot\sigma^3\cdot e^{-3\gamma(h_1-h_2)}\cdot\Phi(\gamma)/\lambda^6$, where $\Phi(\gamma)$ – the function, for γ> 0, has only positive values and does not vanish, it follows from this that this system of equations has a unique solution.

We find the functions $H_k^{(1)}(\lambda,\mu)$, $\tilde{H}_k^{(1)}(\lambda,\mu)$ and $H_k^{(2)}(\lambda,\mu)$ through $B_{k,m}(\lambda)$.

To satisfy the coupling conditions at the boundary of the half-space and the pipe, we then equate ρ=R_1 in (8) and (9). In (8) we decompose the basic solutions $\vec{u}_k^{(+)}$ using [12, formula (8)], turning them into solutions $\vec{R}_{k,m}$. The resulting vector, as well as vector (9), for ρ=R_1, we substitute in (4). So we get three infinite systems of equations for the coupling of half-space and the pipe in displacements. This will fulfill condition (5).

To fulfill the boundary conditions on the inner surface of the pipe, we find the stresses for (9) and equate, at ρ=R_2, the given $\vec{F}_R^0(\phi,z)$, represented by the integral and the Fourier series.

Having received 9 infinite equations, instead of $H_k^{(2)}(\lambda,\mu)$, we substitute the previously expressed functions through $B_{k,m}(\lambda)$, free ourselves from the series in m and the integrals in λ. As a result, we get a set of nine infinite systems of linear algebraic equations for determining unknowns $B_{k,m}(\lambda)$, $A_{k,m}(\lambda)$ and $\tilde{A}_{k,m}(\lambda)$. These infinite systems have the properties of equations of the second kind and, as a consequence, the reduction method can be applied to them.

Having solved this system of equations, we find the unknowns $A_{k,m}(\lambda)$, $\tilde{A}_{k,m}(\lambda)$ and $B_{k,m}(\lambda)$.

Found from the infinite system of equations $B_{k,m}(\lambda)$, we substitute in the expressions for

$H_k^{(1)}(\lambda,\mu)$, $\widetilde{H}_k^{(1)}(\lambda,\mu)$ and $H_k^{(2)}(\lambda,\mu)$. This will determine all unknown problems.

Numerical Studies of the Stressed State

Имеется упругое изотропное полупространство, в котором, параллельно его поверхности, расположена круглая цилиндрическая толстостенная труба. С поверхностью полупространства жестко сцеплен слой. Материал слоя – асфальтобетон, коэффициент Пуассона $\sigma_1 = 0.1$, модуль упругости E_1=140 кН/см^2. Полупространство – щебень и гравий укрепленные цементом, коэффициент Пуассона $\sigma_2 = 0.25$, модуль упругости E_2=90 кН/см^2. Труба – сталь, коэффициент Пуассона σ_3 = 0.25, модуль упругости E_3=20000 кН/см^2. Наружный радиус трубы R_1=30см., внутренний R_2=20 см. Расстояние от верхней границы слоя к центру трубы h_2=45см. Толщина слоя h_1=10см.

With the weight of the processing equipment taken into account, on the upper boundary of the layer, the stresses

$\tau_{yx}^{(h)}(x,z) = -10^8 \cdot (z^2+10^2)^{-2} \cdot (x^2+10^2)^{-2}$,

$\sigma_y^{(h)} = \tau_{yz}^{(h)} = 0$ are given. On the inner surface of the tube, there are no stresses $\sigma_\rho^{(p)} = \tau_{\rho\phi}^{(p)} = \tau_{\rho z}^{(p)} = 0$.

A finite system of equations of order m = 8 was solved. The accuracy of the fulfillment of the boundary conditions for the indicated values of geometric parameters was equal to 10^{-3}.

In Fig. 1. stresses are presented on the upper and lower boundary of the layer along the x axis, at z = 0 in kN/cm^2.

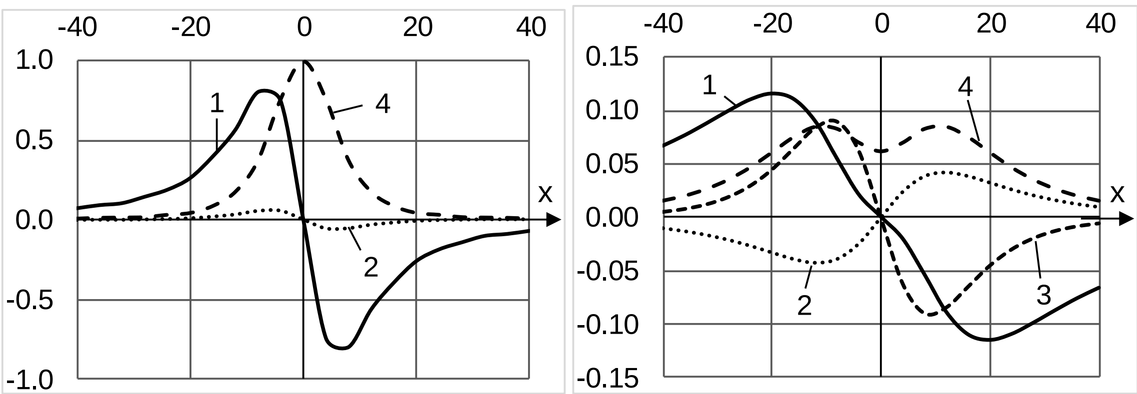

Fig. 1. Stresses at the boundaries of the layer, at z = 0: a - at the upper boundary ($y_1 = h_1$); b - at the lower boundary (y_1 =0); 1 –σ_x; 2 – σ_z; 3 – σ_y; 4 – τ_{xy}

For given tangential stresses τ_{xy} (Fig. 1a, line 4), significant normal stresses σ_x arise at the upper boundary (Fig. 1a, line 1), which decrease at the lower boundary, while remaining maximum. Stresses σ_y also appear at the lower boundary of the layer (Fig. 1b, line 4), although they are set equal to zero at the upper boundary. Stresses σ_z at the upper and lower boundary of the layer do not differ significantly.

In fig. Figure 2 shows the stresses on the pipe surfaces along the radii R_1 and R_2, at z = 0 in kN/cm^2.

The largest stresses that occur on the outer surface of the pipe are normal stresses σ_ϕ (Fig. 2a, line 1), which at ϕ=0.98 have a negative extreme value σ_ϕ= –0.084 kN/cm^2, at ϕ = 2.16 a positive extreme value σ_ϕ= +0.084 kN/cm^2. Small stresses σ_ρ also appear on the outer surface of the pipe in the upper zone (Fig. 2a, line 3).

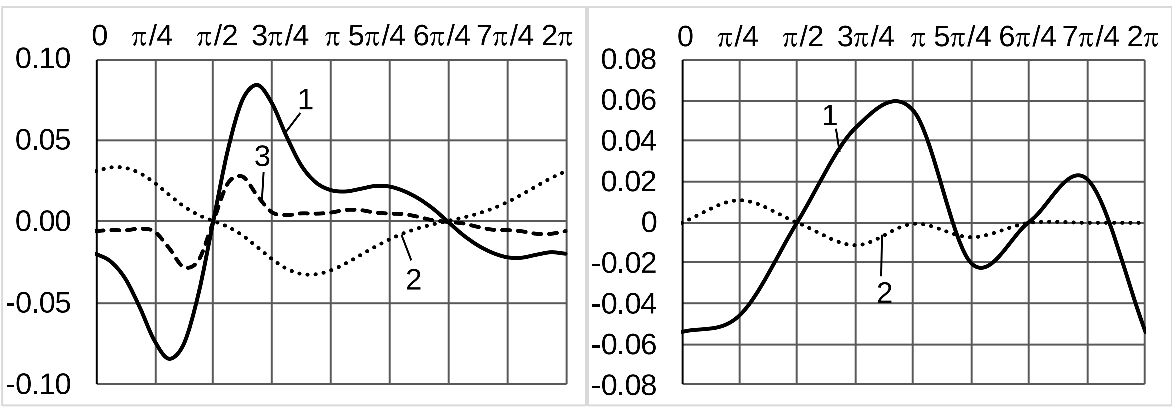

Fig. 2. Stresses on the pipe surfaces, at z = 0: a –on the outer surface; b - on the inner surface; 1 –σ_ϕ; 2 –σ_z; 3 –σ_ρ

On the inner surface, the stresses partially decrease (Fig. 2*b*).

Conclusions

The three-dimensional problem of the theory of elasticity for a multilayer medium consisting of a layer, half-space and a thick-walled pipe, which are interconnected by conjugation conditions, is solved. At the free boundary of the layer and the inner surface of the pipe, stresses are specified.

The proposed solution method is based on the generalized Fourier method and allows determining the stress-strain state of the medium under study with a predetermined accuracy.

Numerical studies were carried out for given nonzero tangential stresses on the layer surface. The analysis showed that the greatest attention should be paid to the normal stresses σ_x in the layer and σ_ϕ in the pipe.

The presented stress state graphs can be used to select geometric characteristics during the design of tunnels and underground utilities.

Further research is relevant for more pipes.

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