ФИЗИКО-МАТЕМАТИЧЕСКИЕ НАУКИ

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VECTOR-VALUED GENERALIZATION OF CONTINUOUS FRAMES AND THEIR NOETHERIAN PERTURBATIONS

Summary. Vector-valued generalization of continuous frames in Banach spaces is considered in this paper. The concepts of $c\tilde{X}$-frame, $c\tilde{X}$-Riesz basis, Banach $c\tilde{X}$-frame and $c\tilde{X}$-atomic decomposition are introduced. Criteria for $c\tilde{X}$-frames, $c\tilde{X}$-Riesz bases, Banach $c\tilde{X}$-frames are found and the relationship between them is established. The stability of $c\tilde{X}$-frame and related (in some sense) $c\tilde{X}$-atomic decompositions, as well as Noetherian perturbations of $c\tilde{X}$-atomic decompositions are also studied.

1. Introduction

The concept of frames in Hilbert spaces has been introduced by R.J. Duffin and A.C. Schaeffer in 1952 [1] in the study of non-harmonic Fourier series with respect to perturbed exponential systems. In the same work, R.J. Duffin and A.C. Schaeffer introduced the concept of abstract frame and extended many of their results to this concept. The interest to frames has grown significantly in the 1980s due to wide applications of wavelet methods in various fields of natural science (see, e.g., N.M. Astafyeva [2], I.M.Dremin, O.V. Ivanov, V.A. Nechitailo [3], etc). For theoretical aspects of this theory we refer the readers to Ch. Chui [4], Y. Meyer [5], I. Daubechies [6], S. Mallat [7], R. Young [8], Ch. Heil [9], O. Christensen [10, 11], etc.

Frames draw growing interest also from a theoretical point of view. As an example, we can mention the connection between the theory of frames and the well-known problem of Kadison and Singer (1959). Modified, but equivalent forms of this problem have been studied in different branches of mathematics such as theory of frames, theory of operators, time-frequency analysis, etc. (for more details see [12-14]). It should be noted that the advantage of Hilbert frames is that every element of a Hilbert space has a frame expansion. This expansion may not be unique. A frame defines a conjugate frame which generates the frame expansion. Moreover, the sequence of coefficients of this expansion has the least l_2 norm. Therefore, the matter of finding new frames is of special scientific interest.

The methods of perturbation theory for linear operators are widely used for establishing the frames. O. Christensen [10, 11] thoroughly studied this matter in case where the perturbations are caused by the compact operators. The case of most general perturbations, i.e. the case of perturbations caused by Noetherian operators, has been studied by B.T. Bilalov and F.A. Guliyeva [15, 16]. The stability of frames in Hilbert spaces was investigated in [17, 18].

The concept of frames in Banach spaces was first treated by K. Gröchenig in [19], where the concepts of Banach frame and atomic decomposition were introduced. Banach frames, atomic decompositions and their stability have also been studied in [20-22], while [15] treated their Noetherian perturbations.

Later, the concept of frame has been generalized in many directions, and this tendency is still going on. In [23], the concepts of g-frame and g-Riesz basis in Hilbert space have been introduced, their basic properties and the relationship between them have been established. g-frames have also been studied in [24-26]. In [16], the concept of t -frame in tensor products of Hilbert spaces has been introduced. The concept of p -frame (a generalization of a frames in Banach spaces) has been introduced and studied in [27] (see also [28, 29]). In a more general case of Banach space of sequences with a canonical basis, p -frames have been studied in [30]. Another generalization of frames in Hilbert spaces is a continuous frame treated in [31] for locally compact space with Radon measure. Continuous frames have been also studied in [32]. Generalizations of the results of [23] to continuous frames in Hilbert spaces can be found in [33].

In this work, we consider a vector-valued generalization of continuous frames in Banach spaces. We introduce the concepts of $c\tilde{X}$-frame, $c\tilde{X}$-Riesz basis and $c\tilde{X}$-atomic decomposition in Banach spaces. We obtain the results concerning $c\tilde{X}$-frameness and $c\tilde{X}$-Riesz basicity, and establish the relationship between them. We also study the stability of $c\tilde{X}$-frame and Noetherian perturbations of $c\tilde{X}$-atomic decomposition.

2. Needful Information

In this section, we give some notations and auxiliary facts.

Throughout this work, X and Z will denote the Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively. X^* will be the space conjugate to X, and the value of the functional $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*). The mapping $\pi_X: X \to X^{**}$ defined by the formula $(x^*, \pi(x)) = (x, x^*)$ is called a canonical mapping. For reflexive space X, canonical mapping $\pi_X: X \to X^{**}$ is an isometric isomorphism. By $L(X, Z)$ we denote the Banach space of linear bounded operators $T: X \to Z$. The kernel and the image of the operator $T \in L(X, Z)$ are denoted by $ker\, T$ and $Im\, T$, respectively. The conjugate of the operator T is denoted by T^*. $T^* \in L(Z^*, X^*)$ and $\|T^*\| = \|T\|$. The operator

$T \in L(X, Z)$ is said to be Noetherian if $ker\, T$, $ker\, T^*$ are finite dimensional and $Im\, T$ is closed.

We will need the following well-known facts.

Theorem 2.1 ([34]). Let $T \in L(X, Z)$. Then the conjugate operator T^* is surjective only when T has a bounded inverse in $Im\, T$.

Theorem 2.2 ([10]). Let $T: X \to X$ be a linear operator. Assume there exist the numbers $\lambda_1, \lambda_2 \in [0,1)$ such that $\|x - Tx\|_X \le \lambda_1 \|x\|_X + \lambda_2 \|Tx\|_X$ for every $x \in X$. Then the operator T is bounded and boundedly invertible in X and $\frac{1-\lambda_2}{1+\lambda_1}\|x\|_X \le \|T^{-1}x\|_X \le \frac{1+\lambda_2}{1-\lambda_1}\|x\|_X$.

Let Ω be some set, $\tilde{X}$ and $\tilde{Z}$ be Banach spaces of vector-valued mappings $\tilde{x} = x(\cdot)$, $x(\omega): \Omega \to X$ and $\tilde{z} = z(\cdot)$, $z(\omega): \Omega \to Z$, respectively. We say that the space $\tilde{X}$ is normally subordinate to the space $\tilde{Z}$ if, for $x(\omega): \Omega \to X$ and $z(\cdot) \in \tilde{Z}$, it follows from $\|x(\omega)\|_X \le \|z(\omega)\|_Z$, $\forall \omega \in \Omega$, that $\tilde{x} = x(\cdot) \in \tilde{X}$ and $\|x(\cdot)\|_{\tilde{X}} \le \|z(\cdot)\|_{\tilde{Z}}$.

The concept below is a generalization of a continuous frame in a Banach space.

Definition 2.1. The mapping $F: \Omega \to L(Z, X)$ is called a $c\tilde{X}$-frame in Z with respect to Ω if $F(\cdot)z \in \tilde{X}$, $\forall z \in Z$ and $\exists A, B > 0$ such that

$$A\|z\|_Z \le \|F(\cdot)z\|_{\tilde{X}} \le B\|z\|_Z, \ \forall z \in Z. \tag{2.1}$$

The constants A and B are called the lower and upper bounds of $c\tilde{X}$-frame, respectively. In case where the right-hand side inequality in (2.1) is true, $F: \Omega \to L(Z, X)$ is said to be $c\tilde{X}$-Besselian in Z with respect to Ω with the bound B. Next, $c\tilde{X}$-frame in Z with respect to Ω will be called simply $c\tilde{X}$-frame in Z. If there exists an operator $S \in L(\tilde{X}, Z)$ such that $S(F(\cdot)z) = z$, $\forall z \in Z$, then the pair (F, S) will be called a Banach $c\tilde{X}$-frame in Z with the bounds A and B, and the operator S will be called a $c\tilde{X}$-frame operator of the mapping F.

Example 2.1. Let $\boldsymbol{X}$ be a Banach space, (Ω, μ) be a measurable space, $Z = l_2$ and $\tilde{X} = L_p(\Omega, \mu, X)$, $1 < p < +\infty$. Assume that $\alpha_n: \Omega \to X$ is such that $\exists A(\omega), B(\omega) \in L_p(\Omega, \mu)$: $\forall \omega \in \Omega$

$$A(\omega)\|\{c_n\}\|_{l_2} \le \|\textstyle\sum_n c_n \alpha_n(\omega)\|_X \le B(\omega)\|\{c_n\}\|_{l_2}, \ \forall \{c_n\} \in l_2. \tag{2.2}$$

Let the mapping $F: \Omega \to L(l_2, X)$ be defined by the formula $F(\omega)\{c_n\}_{n\in N} = \sum_{n=1}^{\infty} c_n \alpha_n(\omega)$. Then F is a $cL_p(\Omega, \mu, X)$ -frame in l_2.

In fact, $\forall \{c_n\}_{n\in N} \in l_2$ we have

$$\|F(\cdot)\{c_n\}_{n\in N}\|^p_{L_p(\Omega,\mu,X)} = \int_\Omega \|\textstyle\sum_{n=1}^{\infty} c_n \alpha_n(\omega)\|^p \, d\mu.$$

Using (2.2), we obtain

$$\|A(\cdot)\|_{L_p(\Omega,\mu)}\|\{c_n\}_{n\in N}\|_{l_2} \le \|F(\cdot)(\{c_n\}_{n\in N})\|_{L_p(\Omega,\mu,X)} \le \|B(\cdot)\|_{L_p(\Omega,\mu)}\|\{c_n\}_{n\in N}\|_{l_2}.$$

Example 2.2. Let $\boldsymbol{Z}$ be a Banach space, $X = l_p$, $1 < p < +\infty$, (Ω, μ) be a measurable space, $\tilde{X} = l_p(\Omega, \mu)$ be a Banach space of sequences $\{a_n(\omega)\}_{n\in N}$, $\omega \in \Omega$, of measurable functions in Ω equipped with the norm $\|\{a_n\}_{n\in N}\|_{l_p(\Omega,\mu)} = \left(\sum_{n=1}^{\infty} \int_\Omega |a_n(\omega)|^p d\mu\right)^{\frac{1}{p}}$. Assume that $\alpha_n: \Omega \to Z^*$ is such that $\forall \omega \in \Omega$ $\{\alpha_n(\omega)\}_{n\in N}$ is a p -frame in Z (see [27]) with the bounds $A(\omega), B(\omega) \in L_p(\Omega, \mu)$, i.e. $\forall \omega \in \Omega$

$$A(\omega)\|z\|_Z \le \left(\textstyle\sum_{n=1}^{\infty} |(z, \alpha_n(\omega))|^p\right)^{\frac{1}{p}} \le B(\omega)\|z\|_Z, \forall z \in Z. \tag{2.3}$$

Define the mapping $F: \Omega \to L(Z, l_p)$ by the formula $F(\omega)z = \{(z, \alpha_n(\omega))\}_{n\in N}$, $\omega \in \Omega$. From (2.3) we have

$$\|F(\omega)z\|_{l_p} = \left(\textstyle\sum_{n=1}^{\infty} |(z, \alpha_n(\omega))|^p\right)^{\frac{1}{p}} \le B(\omega)\|z\|_Z, \ \omega \in \Omega, \ z \in Z.$$

The mapping F is a $cl_p(\Omega, \mu)$ -frame in Z with the bounds $\|A\|_{L_p(\Omega,\mu)}$ and $\|B\|_{L_p(\Omega,\mu)}$. In fact, as $\forall z \in Z$

$$\|F(\cdot)z\|^p_{l_p(\Omega,\mu)} = \left(\textstyle\sum_{n=1}^{\infty} \int_\Omega |(z, \alpha_n(\omega))|^p d\mu\right)^{\frac{1}{p}},$$

taking into account (2.3) we obtain

$$\|A(\cdot)\|_{L_p(\Omega,\mu)}\|z\|_Z \le \|F(\cdot)z\|_{l_p(\Omega,\mu)} \le \|B(\cdot)\|_{L_p(\Omega,\mu)}\|z\|_Z.$$

Remark 2.1. If the mapping $F: \Omega \to L(Z, X)$ is a $c\tilde{X}$-frame in Z, then the operator $U \in L(Z, \tilde{X})$ defined by the formula

$$U(z) = F(\cdot)z, z \in Z, \quad (2.3)$$

is boundedly invertible in $Im\,U$. Therefore, if the mapping $F: \Omega \to L(Z,X)$ is a $c\tilde{X}$-frame in Z, then Z is isomorphic to some subspace of $\tilde{X}$.

The following concept generalizes the one of atomic decomposition in Banach spaces.

Definition 2.2. Let $F: \Omega \to L(Z,X)$ and $\Lambda: \Omega \to L(X,Z)$. The pair (F,Λ) is called a $c\tilde{X}$-atomic decomposition in Z with respect to Ω if

1) $\forall z \in Z$, $\forall f \in Z^*$ $F(\cdot)z \in \tilde{X}$;
2) $\exists A, B > 0$ such that $A\|z\|_Z \le \|F(\cdot)z\|_{\tilde{X}} \le B\|z\|_Z$, $\forall z \in Z$;
3) $\forall z \in Z$ $(z,f) = (F(\cdot)z, f\Lambda(\cdot))$, $\forall z \in Z$, $\forall f \in Z^*$.

The constants A and B are called the lower and upper bounds of $c\tilde{X}$-atomic decomposition (F,Λ), respectively.

3. $c\tilde{X}$-Frames In Banach Spaces

In this section, we give criteria for $c\tilde{X}$-frameness and $c\tilde{X}$-Riesz basicity of a mapping, and establish the relationship between them.

The theorem below presents a characterization of $c\tilde{X}$-Besselian mappings.

Theorem 3.1. Let the mapping $F: \Omega \to L(Z,X)$ be such that $F(\cdot)z \in \tilde{X}$, $\forall z \in Z$. Then F is $c\tilde{X}$-Besselian in Z with a bound B only when there exists an operator $T: \tilde{X}^* \to Z^*$ defined by the formula

$$(z, T\tilde{x}^*) = (F(\cdot)z, \tilde{x}^*),\ \tilde{x}^* \in \tilde{X}^*,\ z \in Z, \quad (3.1)$$

and $\|T\| \le B$.

Proof. Let the mapping F be $c\tilde{X}$-Besselian in Z with a bound B. Then there exists a bounded operator $U: Z \to \tilde{X}$ defined by the formula (2.3). Let's find its conjugate U^*. For $\forall \tilde{x}^* \in \tilde{X}^*$ and $\forall z \in Z$ we have

$$(z, U^*\tilde{x}^*) = (Uz, \tilde{x}^*) = (F(\cdot)z, \tilde{x}^*). \quad (3.2)$$

Consequently, $T = U^*$ and $\|T\| = \|U\| \le B$.

Conversely, let there exist a bounded operator $T: \tilde{X}^* \to Z^*$ defined by the formula (3.1) and $\|T\| \le B$. We have

$$\|F(\cdot)z\|_{\tilde{X}} = \sup_{\|\tilde{x}^*\|=1} |(F(\cdot)z, \tilde{x}^*)| = \sup_{\|\tilde{x}^*\|=1} |(z, T\tilde{x}^*)| \le \|T\|\|z\|_Z \le B\|z\|_Z,$$

i.e. the mapping F is $c\tilde{X}$-Besselian in Z with a bound B. Theorem is proved.

The theorem below presents a criterion for $c\tilde{X}$-frameness of a mapping.

Theorem 3.2. Let the mapping $F: \Omega \to L(Z,X)$ be such that $F(\cdot)z \in \tilde{X}$, $\forall z \in Z$. Then F is a $c\tilde{X}$-frame in Z with a bound B only when there exists a bounded operator $T: \tilde{X}^* \to Z^*$ defined by the formula (3.1) and $Im\,T = Z^*$.

Proof. Let F be a $c\tilde{X}$-frame in Z and the operator U be defined by the formula (2.3). Then U is boundedly invertible in $Im\,U$, and therefore, by Theorem 2.1, the operator U^* maps $\tilde{X}^*$ into Z^*. By Theorem 3.1, the operator T is bounded and from (3.2) we have $T = U^*$. Therefore, $Im\,T = Z^*$.

Conversely, let the operator T defined by the formula (3.1) be bounded and $Im\,T = Z^*$. By Theorem 3.1, the mapping F is $c\tilde{X}$-Besselian in Z. As $T = U^*$, we have $Im\,U^* = Z^*$. Consequently, by Theorem 2.1, the operator U is boundedly invertible in $Im\,U$, i.e. F is a $c\tilde{X}$-frame in Z.

The concept below is a generalization of a Riesz basis.

Definition 3.1. A mapping $F: \Omega \to L(Z,X)$ is called a $c\tilde{X}^*$-Riesz basis for Z^* with respect to Ω if

1) $F(\cdot)z = 0$ implies $z = 0$;
2) there exists the operator $T: \tilde{X}^* \to Z^*$ defined by the formula (3.1) and $\exists A, B > 0$ such that

$$A\|\tilde{x}^*\|_{\tilde{X}^*} \le \|T\tilde{x}^*\|_{Z^*} \le B\|\tilde{x}^*\|_{\tilde{X}^*},\ \forall \tilde{x}^* \in \tilde{X}^*. \quad (3.3)$$

The constants A and B are called the bounds of $c\tilde{X}^*$-Riesz basis.

Let's establish the relationship between a $c\tilde{X}^*$-Riesz basis and a $c\tilde{X}$-frame in Z.

Theorem 3.3. Let Z be a reflexive Banach space and the mapping $F: \Omega \to L(Z,X)$ be a $c\tilde{X}^*$-Riesz basis for Z^* with the bounds A and B. Then $F: \Omega \to L(Z,X)$ is a $c\tilde{X}$-frame in Z with the bounds A and B.

Proof. Let $F: \Omega \to L(Z,X)$ be a $c\tilde{X}^*$-Riesz basis for Z^* with the bounds A and B. It follows from the inequality (3.3) that the bounded operator T is injective and $Im\,T$ is closed. In fact, if $T\tilde{x}^* = 0$, then from the inequality on the left-hand side of (3.3) we obtain

$$\|\tilde{x}^*\|_{\tilde{X}^*} = 0, \text{ i.e. } \tilde{x}^* = 0. \text{ Let } f_n = T\tilde{x}_n^* \text{ и } f = \lim_{n\to\infty} f_n.$$

Then

$$A\|\tilde{x}_n^* - \tilde{x}_m^*\|_{\tilde{X}^*} \le \|f_n - f_m\|_{Z^*} \to 0$$

as $n, m \to \infty$. Therefore, in view of the completeness of $\tilde{X}^*$, there exists $\tilde{x}^* = \lim_{n\to\infty} \tilde{x}_n^*$. By virtue of the continuity of the operator T, we have $T\tilde{x}^* = \lim_{n\to\infty} T\tilde{x}_n^* = f$. Let's show that $Im\, T = Z^*$. Assume the contrary, i.e. assume $Im\, T \neq Z^*$. The reflexivity of Z implies the existence of $z \in Z$, $z \neq 0$ such that $(z, T\tilde{x}^*) = 0$ for $\forall \tilde{x}^* \in \tilde{X}^*$. Consequently, $(F(\cdot)z, \tilde{x}^*) = 0$. Hence, $F(\cdot)z = 0$ and $z = 0$. This contradicts the condition $z \neq 0$, therefore $Im\, T = Z^*$. Thus, the operator T is boundedly invertible. It is clear that $\|T\|$ and $\|T^{-1}\|^{-1}$ are the bounds of the $c\tilde{X}^*$-Riesz basis F. As $U^* = T$, the operator U is boundedly invertible, and, consequently, F is a $c\tilde{X}$-frame in Z with the bounds

$$\|U\| = \|T^*\| = \|T\|, \|U^{-1}\|^{-1} = \|(T^*)^{-1}\|^{-1} = \|T^{-1}\|^{-1}.$$

Theorem is proved.

Now we consider the conditions under which a $c\tilde{X}$-frame becomes a $c\tilde{X}^*$-Riesz basis. The following theorem is true.

Theorem 3.4. Let $\tilde{X}$ be a reflexive space, the mapping $F: \Omega \to L(Z, X)$ be a $c\tilde{X}$-frame in Z and the operator U be defined by the formula (2.3). The following properties are equivalent:

1) F is a $c\tilde{X}^*$-Riesz basis for Z^*;

2) the operator T defined by (3.1) is injective;

3) $Im\, U = \tilde{X}$.

Proof. 1) $\Rightarrow$ 2) is obvious.

Prove that 2) $\Rightarrow$ 3). Let the operator T be injective. By Theorem 3.2, the operator T is surjective. Then it is boundedly invertible. Using Remark 2.1, from the reflexivity of the space $\tilde{X}$ we obtain the reflexivity of the space Z. Let's show the validity of the relation $U = \pi_{\tilde{X}}^{-1} T^* \pi_Z$. In fact, $\forall \tilde{x}^* \in \tilde{X}^*$ and $\forall z \in Z$ we have

$$(Uz, \tilde{x}^*) = (z, T\tilde{x}^*) = (T\tilde{x}^*, \pi_Z(z)) = (\tilde{x}^*, T^*\pi_Z(z)).$$

On the other hand, we have $(Uz, \tilde{x}^*) = (\tilde{x}^*, \pi_{\tilde{X}}(Uz))$. Therefore, $\pi_{\tilde{X}} U = T^*\pi_Z$, i.e. $U = \pi_{\tilde{X}}^{-1} T^* \pi_Z$. Hence we conclude that the operator U is also boundedly invertible. Consequently, $Im\, U = \tilde{X}$.

Prove that 3) $\Rightarrow$ 1). By condition 3), the operator U is boundedly invertible. Consequently, $U^* = T$ implies that the operator T is boundedly invertible. Therefore, F is a $c\tilde{X}^*$-Riesz basis for Z^*. Theorem is proved.

Theorem 3.5. Let the mapping $F: \Omega \to L(Z, X)$ be a $c\tilde{X}$-frame in Z and the operator U be defined by the formula (1.3). The following properties are equivalent:

1) $Im\, U$ is complementable in $\tilde{X}$;

2) there exists $S \in L(\tilde{X}, Z)$ such that (F, S) is a Banach $c\tilde{X}$-frame in Z;

3) there exists $G \in L(Z^*, \tilde{X}^*)$ such that $(z, f) = (F(\cdot)z, Gf)$ $\forall f \in Z^*$, $\forall z \in Z$.

Proof. 1) $\Leftrightarrow$ 2). Let $Im\, U$ be complementable in $\tilde{X}$. Then there exists a projector $P: \tilde{X} \to Im\, U$. As the mapping $F: \Omega \to L(Z, X)$ forms a $c\tilde{X}$-frame in Z, the operator U has a bounded inverse U^{-1} in $Im\, U$. Let D be an arbitrary bounded continuation of U^{-1} by $Im\, U$ to the whole of $\tilde{X}$. Consider the operator $S = DP$. It is clear that $S \in L(\tilde{X}, Z)$ and $SU = I$, i.e. (F, S) is a Banach $c\tilde{X}$-frame in Z.

Now let (F, S) be a Banach $c\tilde{X}$-frame in Z. Consider the operator $P = US$. Then, in view of $SU = I$, we have $P^2 = USUS = US = P$, and, consequently, P is a projector in $\tilde{X}$. Let's show that $Im\, P = Im\, U$. It is clear that $Im\, P \subset Im\, U$. Let $\tilde{x} \in Im\, U$ and $Uz = \tilde{x}$. Then $z = S\tilde{x}$, and, consequently, $Uz = US\tilde{x} = P\tilde{x}$, i.e. $Im\, P = Im\, U$. Thus, P is a projector from $\tilde{X}$ to $Im\, U$. Hence, $Im\, U$ is a complementable subspace in $\tilde{X}$.

2) $\Leftrightarrow$ 3). Define the operator G as follows $G = S^*$. For $\forall z \in Z$ and $\forall f \in Z^*$ we have $(z, f) = (SF(\cdot)z, f) = (F(\cdot)z, S^*f) = (F(\cdot)z, Gf)$.

Conversely, define the operator $S: \tilde{X} \to Z$ by the formula $(S\tilde{x}, f) = (\tilde{x}, Gf)$ for $\forall \tilde{x} \in \tilde{X}$ and $\forall f \in Z^*$. Then $|(S\tilde{x}, f)| \leq \|\tilde{x}\|_{\tilde{X}} \|G\| \|f\|$. Hence $\|S\| \leq \|G\|$. In what follows

$$(SF(\cdot)z, f) = (F(\cdot)z, Gf) = (z, f).$$

Consequently, $S(F(\cdot)z) = z$, $\forall z \in Z$. Theorem is proved.

4. Stability And Noetherian Perturbations Of $c\tilde{X}$-Frames

In this section, we consider the stability of $c\tilde{X}$-frames and $c\tilde{X}$-atomic decompositions as well as their Noetherian perturbations.

The theorem below concerns the stability of $c\tilde{X}$-frames in Banach spaces.

Theorem 4.1. Let $F: \Omega \to L(Z, X)$ be a $c\tilde{X}$-frame in Z with the bounds A and B. Assume that the mapping $G: \Omega \to L(Z, X)$ is such that $G(\cdot)z \in \tilde{X}$, there exist the numbers $\lambda, \beta, \mu \geq 0$ such that the following conditions hold:

1) $max\left\{\lambda + \frac{\mu}{A}, \beta\right\} < 1$;

2) $\|F(\cdot)z - G(\cdot)z\|_{\tilde{X}} \leq \lambda \|F(\cdot)z\|_{\tilde{X}} + \beta \|G(\cdot)z\|_{\tilde{X}} + \mu \|z\|_Z$.

Then G is a $c\tilde{X}$-frame in Z with the bounds $\frac{A(1-\lambda)-\mu}{1+\beta}$ and $\frac{B(1+\lambda)+\mu}{1-\beta}$.

Proof. By condition 2), $\forall z \in Z$ we have

$$\|G(\cdot)z\|_{\tilde{X}} \leq \|F(\cdot)z\|_{\tilde{X}} + \|F(\cdot)z - G(\cdot)z\|_{\tilde{X}} \leq (1+\lambda)\|F(\cdot)z\|_{\tilde{X}} + \beta\|G(\cdot)z\|_{\tilde{X}} + \mu\|z\|_Z.$$

Hence,

$$(1-\beta)\|G(\cdot)z\|_{\tilde{X}} \leq (1+\lambda)\|F(\cdot)z\|_{\tilde{X}} + \mu\|z\|_Z,$$

or

$$\|G(\cdot)z\|_{\tilde{X}} \leq \frac{1+\lambda}{1-\beta}\|F(\cdot)z\|_{\tilde{X}} + \frac{\mu}{1-\beta}\|z\|_Z.$$

Then, using (2.1), we have

$$\|G(\cdot)z\|_{\tilde{X}} \leq \frac{B(1+\lambda)+\mu}{1-\beta}\|z\|_Z.$$

We need to establish a left-hand side $c\tilde{X}$-frame inequality for the mapping G. We have

$$\|G(\cdot)z\|_{\tilde{X}} \geq \|F(\cdot)z\|_{\tilde{X}} - \|F(\cdot)z - G(\cdot)z\|_{\tilde{X}} \geq (1-\lambda)\|F(\cdot)z\|_{\tilde{X}} - \beta\|G(\cdot)z\|_{\tilde{X}} - \mu\|z\|_Z,$$

or

$$(1+\beta)\|G(\cdot)z\|_{\tilde{X}} \geq (1-\lambda)\|F(\cdot)z\|_{\tilde{X}} - \mu\|z\|_Z.$$

Consequently,

$$\|G(\cdot)z\|_{\tilde{X}} \geq \frac{1-\lambda}{1+\beta}\|F(\cdot)z\|_{\tilde{X}} - \frac{\mu}{1+\beta}\|z\|_Z \geq \frac{A(1-\lambda)-\mu}{1+\beta}\|z\|_Z.$$

Thus, G is a $c\tilde{X}$-frame in Z. Theorem is proved.

The next theorem concerns the stability of a Banach $c\tilde{X}$-frame.

Theorem 4.2. Let (F, S) be a Banach $c\tilde{X}$-frame in Z, the mapping $G: \Omega \to L(Z, X)$ be such that $G(\cdot)z \in \tilde{X}$ and the operators $U, V: Z \to \tilde{X}$ be defined by the equalities $U(z) = F(\cdot)z$, $V(z) = G(\cdot)z$, respectively. Assume that there exist the numbers $\lambda, \beta \in [0; 1)$, $\mu \geq 0$ such that the following conditions hold:

1) $\lambda\|US\| + \beta\|VS\| + \mu\|S\| < 1$;
2) $\|F(\cdot)z - G(\cdot)z\|_{\tilde{X}} \leq \lambda\|F(\cdot)z\|_{\tilde{X}} + \beta\|G(\cdot)z\|_{\tilde{X}} + \mu\|z\|_Z$.

Then there exists $S_1 \in L(\tilde{X}, Z)$ such that (G, S_1) forms a Banach $c\tilde{X}$-frame in Z.

Proof. It is clear that the mapping F is a $c\tilde{X}$-frame in Z with the bounds $\|S\|^{-1}$ and $\|U\|$, because $\|S\|^{-1}\|z\|_Z = \|S\|^{-1}\|SUz\|_Z \leq \|Uz\|_{\tilde{X}} \leq \|U\|\|z\|_Z$. As $P = US$ is a projector, we have $\|US\| \geq 1$. Then $\lambda + \mu\|S\| \leq \lambda\|US\| + \beta\|VS\| + \mu\|S\| < 1$, and, by Theorem 4.1, the mapping $G: \Omega \to L(Z, X)$ is a $c\tilde{X}$-frame in Z with the bounds

$$\frac{\|S\|^{-1}(1-\lambda)-\mu}{1+\beta} \text{ and } \frac{(1+\lambda)\|U\|+\mu}{1-\beta}.$$

Next, $\forall \tilde{x} \in \tilde{X}$ from condition 2) we obtain

$$\|(U-V)S\tilde{x}\|_{\tilde{X}} \leq \lambda\|US\tilde{x}\|_{\tilde{X}} + \beta\|VS\tilde{x}\|_{\tilde{X}} + \mu\|S\tilde{x}\|_Z \leq (\lambda\|US\| + \beta\|VS\| + \mu\|S\|)\|\tilde{x}\|_{\tilde{X}}.$$

Hence, in view of $\lambda\|US\| + \beta\|VS\| + \mu\|S\| < 1$, we have $\|(U-V)S\| < 1$. Consequently, the operator $I - (U-V)S$ is boundedly invertible. Let $S_1 = S(I - (U-V)S)^{-1}$. It is clear that $S_1 \in L(\tilde{X}, Z)$. Next, $S_1 V = I$. In fact, as

$$V = VSU = U - (U-V)SU = (I - (U-V)S)U,$$

we have

$$S_1 V = S(I - (U-V)S)^{-1}V = S(I - (U-V)S)^{-1}(I - (U-V)S)U = SU = I.$$

Thus, (G, S_1) is a Banach $c\tilde{X}$-frame in Z. Theorem is proved.

Let's state another theorem on the stability of a Banach $c\tilde{X}$-frame.

Theorem 4.3. Let (F, S) be a Banach $c\tilde{X}$-frame in Z, the mapping $G: \Omega \to L(Z, X)$ be such that $G(\cdot)z \in \tilde{X}$ and the operators $U, V: Z \to \tilde{X}$ be defined by the equalities $U(z) = F(\cdot)z$, $V(z) = G(\cdot)z$, respectively. Assume that there exist the numbers $\lambda, \beta \in [0; 1)$, $\mu \geq 0$ that satisfy the following conditions:

1) $\lambda\|US\| + \beta\|I - US\| + \mu\|S\| < 1$;
2) $\|F(\cdot)z - G(\cdot)z\|_{\tilde{X}} \leq \lambda\|F(\cdot)z\|_{\tilde{X}} + \beta\|G(\cdot)z\|_{\tilde{X}} + \mu\|z\|_Z$.

Then there exists $S_1 \in L(\tilde{X}, Z)$ such that (G, S_1) forms a Banach $c\tilde{X}$-frame in Z.

Proof. As in the proof of Theorem 4.2, it follows from condition 2) that the mapping G is a $c\tilde{X}$-frame in

Z. Consider the operator $I-(U-V)S$. By condition 2), $\forall\tilde{x}\in\tilde{X}$ we have

$$\|(U-V)S\tilde{x}\|_{\tilde{X}}\le\lambda\|US\tilde{x}\|_{\tilde{X}}+\beta\|VS\tilde{x}\|_{\tilde{X}}+\mu\|S\tilde{x}\|_{Z}\le$$
$$\le(\lambda\|US\|+\beta\|I-US\|+\mu\|S\|)\|\tilde{x}\|_{\tilde{X}}+\beta\|(I-(U-V)S)\tilde{x}\|_{\tilde{X}}.$$

Hence, according to Theorem 2.2, the operator $I-(U-V)S$ is boundedly invertible, and $S_1=S(I-(U-V)S)^{-1}$ is a required operator. Theorem is proved.

Now we consider the Noetherian perturbations of $c\tilde{X}$-atomic decompositions.

Theorem 4.4. Let the mappings $F:\Omega\to L(Z,X)$ and $\Lambda:\Omega\to L(X,Z)$ be such that (F,Λ) is a $c\tilde{X}$-atomic decomposition in Z with the bounds A and B. Let W be a Banach space, $N\in L(Z,W)$ be a Noetherian operator and $\Gamma(\omega)=N\Lambda(\omega)$, $\omega\in\Omega$. Then there exists a mapping $G:\Omega\to L(W,X)$ such that (G,Γ) is a $c\tilde{X}$-atomic decomposition in $Im\,N$.

Proof. The Noetherianness of the operator N implies that $ker\,N$ is complementable in Z. Let $Z=ker\,N+Z_1$. Denote the restriction of the operator N to Z_1 by N_1. It is clear that the operator N_1 maps Z_1 into $Im\,N$ and $ker\,N_1=\{0\}$. Then the operator N_1 has a bounded inverse N_1^{-1} in $Im\,N$. Let $G(\omega)=F(\omega)N_1^{-1}$, $\omega\in\Omega$. $\forall w\in Im\,N$ we have

$$\|G(\cdot)w\|_{\tilde{X}}=\|F(\cdot)N_1^{-1}w\|_{\tilde{X}}\le B\|N_1^{-1}w\|_Z\le B\|N_1^{-1}\|\|w\|_W,$$

and

$$\|G(\cdot)w\|_{\tilde{X}}=\|F(\cdot)N_1^{-1}w\|_{\tilde{X}}\ge A\|N_1^{-1}w\|_Z\ge A\|N_1\|^{-1}\|w\|_W,$$

i.e. G is a $c\tilde{X}$-frame in $Im\,N$.

Take $\forall h\in W^*$ and $\forall w\in Im\,N$. Using condition 3) of Definition 2.2, we obtain

$$(w,h)=(N_1^{-1}w,hN)=(F(\cdot)N_1^{-1}w,hN\Lambda(\cdot))=(G(\cdot)w,h\Gamma(\cdot)).$$

Thus, (G,Γ) is a $c\tilde{X}$-atomic decomposition in $Im\,N$. Theorem is proved.

Theorem 4.5. Let (F,S) be a Banach $c\tilde{X}$-frame in Z, W be a Banach space, $N\in L(Z,W)$ be a Noetherian operator and $S_1=NS$. Then there exists a mapping $G:\Omega\to L(W,X)$ such that (G,S_1) is a Banach $c\tilde{X}$-frame in $Im\,N$.

Proof. Assume that Z_1 is a complement of $ker\,N$ in Z and N_1 is a restriction of the operator N to Z_1. The operator N_1 has a bounded inverse N_1^{-1} in $Im\,N$. Using the same reasoning as used in the proof of Theorem 4.4, we obtain that the mapping $G:\Omega\to L(Z,X)$ is such that $G(\omega)=F(\omega)N_1^{-1}$, $\omega\in\Omega$, is a $c\tilde{X}$-frame in $Im\,N$. Next, for $\forall w\in Im\,N$ we have

$S_1(G(\cdot)w)=NS(F(\cdot)N_1^{-1}w)=NN_1^{-1}w=w.$

Thus, (G,S_1) is a Banach $c\tilde{X}$-frame in $Im\,N$. Theorem is proved.

Let $\tilde{Y}$ be a Banach space of operator-valued mappings $\tilde{\Lambda}=\Lambda(\cdot)$, $\Lambda(\omega):\Omega\to L(X,Z)$.

Definition 4.1. Mappings $\Lambda:\Omega\to L(X,Z)$ and $\Gamma:\Omega\to L(X,Z)$ are called $\tilde{Y}$-close if $\Lambda(\cdot)-\Gamma(\cdot)\in\tilde{Y}$.

Consider the stability of a $c\tilde{X}$-atomic decomposition for $\tilde{Y}$-close mappings. The following theorem is true.

Theorem 4.6. Let $\tilde{X}^*$ be normally subordinate to $\tilde{Y}$, the mappings $F:\Omega\to L(Z,X)$ and $\Lambda:\Omega\to L(X,Z)$ be such that (F,Λ) is a $c\tilde{X}$-atomic decomposition in Z with the bounds A and B. Let the mapping $\Gamma:\Omega\to L(X,Z)$ be $\tilde{Y}$-close to Λ and $\|\Lambda(\cdot)-\Gamma(\cdot)\|_{\tilde{Y}}<\frac{1}{B}$. Then there exists a mapping $G:\Omega\to L(Z,X)$ such that (G,Γ) is a $c\tilde{X}$-atomic decomposition in Z.

Proof. Since for any $f\in Z^*$ we have

$$\|f\Lambda(\omega)-f\Gamma(\omega)\|\le\|f\|\|\Lambda(\omega)-\Gamma(\omega)\|,\ \omega\in\Omega,$$

and $\tilde{X}^*$ is normally subordinate to $\tilde{Y}$ we obtain

$$\|f\Lambda(\cdot)-f\Gamma(\cdot)\|_{\tilde{X}^*}\le\|f\|\|\Lambda(\cdot)-\Gamma(\cdot)\|_{\tilde{Y}}.$$

Consider the operator $K:Z\to Z$ defined by the formula

$$(Kz,f)=(F(\cdot)z,f\Lambda(\cdot)-f\Gamma(\cdot)),\ \forall f\in Z^*,\ \forall z\in Z.$$

Then

$$\|Kz\|=\sup_{\|f\|=1}|(Kz,f)|=\sup_{\|f\|=1}|(F(\cdot)z,f\Lambda(\cdot)-f\Gamma(\cdot))|\le$$
$$\le\|F(\cdot)z\|_{\tilde{X}}\sup_{\|f\|=1}\|f(\Lambda(\cdot)-\Gamma(\cdot))\|_{\tilde{X}^*}\le B\|\Lambda(\cdot)-\Gamma(\cdot)\|_{\tilde{Y}}\|z\|_Z<\|z\|_Z.$$

Thus, the operator $D=I-K$ is boundedly invertible. Let $G(\omega)=F(\omega)D^{-1}$, $\omega\in\Omega$. Then G forms a $c\tilde{X}$-frame in Z. The last condition in Definition 2.2 implies $(z,f)=(F(\cdot)z,f\Lambda(\cdot))$, $\forall f\in Z^*$, $\forall z\in Z$. Then $(Dz,f)=(F(\cdot)z,f\Gamma(\cdot))$. Next, for $\forall z\in Z$ and $\forall f\in Z^*$we obtain

$$(z, f) = (DD^{-1}z, f) = (F(\cdot)D^{-1}z, f\Gamma(\cdot)) = (G(\cdot)z, f\Gamma(\cdot)).$$

Therefore, (G, Γ) is a $c\tilde{X}$-atomic decomposition in Z. Theorem is proved.

In case where (Ω, μ) is a measurable space, $\tilde{X} = L_p(\Omega, \mu, X)$, $p \in (1; +\infty)$ and $\tilde{Y} = L_q(\Omega, \mu, L(X, Z))$, $\frac{1}{p} + \frac{1}{q} = 1$, Theorem 4.6 has the following corollary.

Corollary 4.1. Let the mappings $F: \Omega \to L(Z, X)$ and $\Lambda: \Omega \to L(X, Z)$ be such that (F, Λ) is a $cL_p(\Omega, \mu, X)$ -atomic decomposition in Z with the bounds A and B, and let the mapping $\Gamma: \Omega \to L(X, Z)$ satisfy the condition

$$\int_\Omega \|\Lambda(\omega) - \Gamma(\omega)\|^q \, d\mu < B^{-q}.$$

Then there exists a mapping $G: \Omega \to L(Z, X)$ such that (G, Γ) is a $cL_p(\Omega, \mu, X)$ -atomic decomposition in Z.

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